Self-Assessment Exercise: Computational Finance

Question 1. In the context of binomial distribution, we consider tossing a coin three times. What is the set of all possible outcomes Ω?

- If $p(H) = p$, $p(T) = q = 1 - p$, and the tosses are independent, what is the probability of each element $\omega \in \Omega$?
- Consider the event A: "The first toss is a head". What are its outcomes and $p(A)$?

Question 2. Consider the space of two coin tosses $\Omega = \{HH, HT, TH, TT\}$, and let stock prices be given by

$$S_0 = 100, S_1(H) = 120, S_1(T) = 80, S_2(HH) = 140, S_2(HT) = 100, S_2(TH) = 100, S_2(TT) = 60.$$

- List all the sets in $\sigma(S_1)$ (σ-algebra generated by S_1)
- Define the random variable
$$X = \begin{cases} 1, & \text{if } S_2 = 100, \\ 0, & \text{if } S_2 \neq 100, \end{cases}$$
list all the sets in $\sigma(X)$.

Question 3. Suppose $X \sim \mathcal{N}(\mu, \sigma)$ and $Y = a + bX$, where a and b are constant, and $b \neq 0$. Determine the expectation $E[Y]$, the variance $Var[Y]$ and the distribution function of Y.

Question 4. A positive random variable X is log-normally distributed with the probability density function

$$f_X(x) = \begin{cases} \frac{1}{x\sigma\sqrt{2\pi}} \exp\left(-\frac{(\ln x - \mu)^2}{2\sigma^2}\right) & \text{for } x > 0, \\ 0 & \text{otherwise.} \end{cases}$$

Calculate the expectation and variance of X as $E[X] = \exp\left(\mu + \frac{\sigma^2}{2}\right)$ and $Var[X] = \Big(\exp(\sigma^2) - 1\Big) \exp(2\mu + \sigma^2)$, respectively.

Question 5. Let $f \in C^6(\mathbb{R})$ and $x, h \in \mathbb{R}$, using the Taylor series show that

$$f''(x) = \frac{1}{h^2}\left\{-\frac{1}{12}f(x+2h) + \frac{4}{3}f(x+h) - \frac{5}{2}f(x) + \frac{4}{3}f(x-h) - \frac{1}{12}f(x-2h)\right\} + \mathcal{O}(h^4)$$

for $h \to 0$.

Question 6. Suppose that the stock price is modelled by

$$S_{j+1} = S_j \cdot \left(1 + r\Delta t + \sigma\sqrt{\Delta t} Z_j\right),$$

where Z_j is the standard normally distributed random variable, $j = 0, \ldots, N$. Write a Python (or Matlab/Julia) function to visualize the trajectories of the stock price for $S_0 = 100$, $\Delta t = 0.01$, $r = 0.02$, $\sigma = 0.1, 0.2, 0.5$.